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Nodal basis functions in p-adaptive finte element methods

Hàm nút cơ sở dùng trong phương pháp phần tử hữu hạn thích nghi loại p

Hieu Nguyen[a,b,*], Quoc Hung Phan[a,b], Tina Mai[a,b]
Nguyễn Trung Hiếu[a,b,*], Phan Quốc Hưng[a,b], Mai Ti Na[a,b]

[a]*Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam*
[b]*Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam*

[a]*Viện Nghiên cứu và Phát triển Công nghệ Cao, Trường Đại học Duy Tân, Đà Nẵng, Việt Nam*
[b]*Khoa Khoa học Tự nhiên, Trường Đại học Duy Tân, Đà Nẵng, Việt Nam*

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Abstract

In this paper, we systematically define, present and prove the main properties of nodal basis functions utilized in p-adaptive finite element methods.

Keywords: Nodal points; Nodal basis functions; p-adaptive finite elements

Tóm tắt

Trong bài báo này, chúng tôi định nghĩa, giới thiệu và chứng minh một cách có hệ thống các tính chất chính của hàm nút cơ sở dùng trong phương pháp phần tử hữu hạn thích nghi loại p.

Từ khóa: Điểm nút; Hàm nút cơ sở; Phần tử hữu hạn loại p

1. Introduction

In adaptive finite element methods, the p-approach uses elements of varying degrees to represent the approximate solution [1, 2, 3]. Nodal basis functions are commonly utilized in this approach [4, 5, 6]. The knowledge about this type of functions is usually considered basic. However, there is currently no literature covering it in detail. This paper attempts to fill the void by sytematically defining, presenting and proving the main properties of nodal basis functions.

2. Nodal points

Let Ω in $\mathbb{R}^2$ be the bounded domain of the partial differential equation we are working with. For simplicity of exposition, we assume that Ω is a polygon. Let $\mathcal{T}$ be a triangulation of Ω, t be an element (triangle) in $\mathcal{T}$. To define the nodal basis functions associated with t, we begin with the definition of *nodal points*.

Definition 2.1. *Nodal points of an element (triangle) t of degree p are:*

(i) three vertex nodal points at the vertices.

[*]*Corresponding Author*: Hieu Nguyen; Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam; Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam.
Email: nguyentrunghieu14@duytan.edu.vn

(ii) *$p-1$ edge nodal points equally spaced in the interior of each edge.*

(iii) *interior nodal points placed at the intersections of lines that are parallel to edges and connecting edge nodal points.*

Nodal points of an element of degree p are sometimes referred to as nodal points of degree p. Note that linear elements ($p = 1$) have only vertex nodal points and quadratic elements ($p = 2$) have only vertex and edge nodal points. Figure 1 shows examples of nodal points for element of degree for $p = 1,\dots,3$. Definition 2.1 above is a descriptive one. Here, we adopt, for practical purposes, the following result using barycentric coordinates.

3. Nodal basis functions

Let $\mathscr{P}_p(t)$ be the space of polynomials of degree equal or less than p, restricted on element t. The canonical basis of $\mathscr{P}_p(t)$ is

$$\{1, x, y, xy, \dots, x^{p-1}y, xy^{p-1}, x^p, y^p\}.$$

This basis is simple but is not convenient to incorporate in finite element methods. In the next few steps, we will prepare for the definition of another basis of $\mathscr{P}_p(t)$ which is usually used in practice.

Lemma 3.1. *Let P be a polynomial of degree $p \geq 1$ that vanishes on the straight line L defined by equation $L(x, y) = 0$. Then we can write $P = LQ$, where Q is a polynomial of degree $p-1$.*

Chứng minh. Make an affine change of coordinates to $(\hat{x}, y)$ such that $L(x, y) = \hat{x}$ (if $L(x, y) = y$ then no change of coordinates is necessary). Let

$$P(\hat{x}, y) = \sum_{i=0}^{p} \sum_{j=0}^{i} c_{ij}\hat{x}^j y^{i-j}. \qquad (1)$$

In the new coordinate system, the equation of L is $\hat{x} = 0$. Since $P|_L \equiv 0$, plugging $\hat{x} = 0$ into equation (1) we have $\sum_{i=0}^{p} c_{i0} y^i \equiv 0$. This implies that $c_{i0} = 0$ for all $i = 0,\dots p$. Therefore,

$$\begin{aligned} P(\hat{x}, y) &= \sum_{i=1}^{p} \sum_{j=1}^{i} c_{ij}\hat{x}^j y^{i-j} \\ &= \hat{x} \sum_{i=0}^{p-1} \sum_{j=0}^{i} \hat{x}^j y^{i-j} \\ &= LQ. \end{aligned}$$

Clearly, Q is a polynomial of degree $p-1$.

Lemma 3.2. *If $P \in \mathscr{P}_p(t)$ vanishes at all of the nodal points of degree p of t, then P is the zero polynomial.*

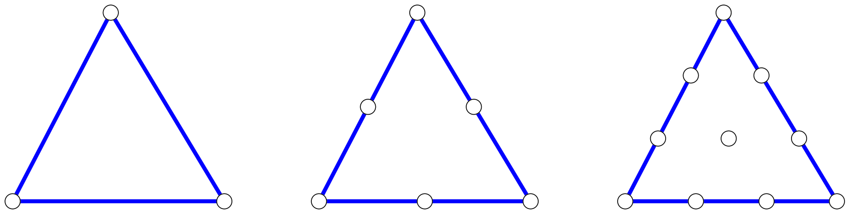
Chứng minh. The proof is by induction on p. Denote v_1, v_2, v_3 and ℓ_1, ℓ_2, ℓ_3 respectively be the vertices and edges of t as shown in Figure 2. In addition, let L_1, L_2, L_3 be the linear functions that define the lines, on which lie the edges ℓ_1, ℓ_2, ℓ_3.

For $p = 1$, P is a linear polynomial that vanishes at two different points v_2 and v_3 of l_1. Therefore, $P|_{\ell_1} \equiv 0$. By Lemma 3.1, $P = cL_1$, where c is a constant (polynomial of degree 0). On the other hand, P equals zero at v_1 and L_1 is nonzero at v_1. This implies that $c = 0$. Hence, $P \equiv 0$.

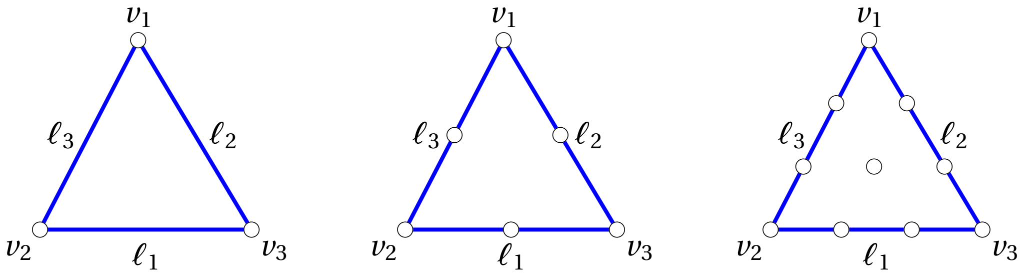
For $p = 2$, P is a quadratic polynomial that vanishes at three different nodal points on ℓ_1. Therefore, $P|_{\ell_1} \equiv 0$. Again by Lemma 3.1, $P = L_1Q$, where Q is a linear function (polynomial of degree 1). Since L_1 is nonzero along ℓ_2 except at v_3, Q needs to be zero at least at two points on ℓ_2: v_1 and the midpoint of l_2. Hence, $Q = cL_2$, where c is a constant. Consequently $P = cL_1L_2$. On the other hand, P needs to be zero at the midpoint of ℓ_3 also. This implies that $c = 0$. Therefore, $P \equiv 0$.

For $p = 3$, using a similar argument, we have $P = cL_1L_2L_3$, where c is a constant. In order for P to be zero at the interior nodal point of degree 3, c needs to be 0. Hence, $P \equiv 0$.

Assume that the lemma holds for polynomials of degree up to p. For $P \in \mathscr{P}_{p+1}(t)$, again by a similar argument for $p = 1, 2, 3$, we know that $P = L_1L_2L_3Q$, where Q is a polynomial of degree $p-3$ or less. Furthermore, Q vanishes at all of the interior nodal points of t. These points can



Hình 1. Nodal points of elements of degree p.



Hình 2. Vertices and edges of elements of degree $p = 1, 2, 3$.

be seen as nodal points of degree $p-3$ of triangle t' laid inside t. Examples for $p = 4, 5, 6$ are illustrated in Figure 3. By induction hypothesis, Q is the zero polynomial. Consequently, P is the zero polynomial.

Now we define nodal basis functions for element t.

Theorem 3.3. *Consider a way of labeling the nodal points of t, an element of degree p, from n_1 to n_{N_p}. Let ϕ_l be the polynomial of degree p that equals 1 at the nodal point n_l and equals 0 at all other nodal points of t. Then $\{\phi_l\}_{l=1}^{N_p}$ is a basis of $\mathscr{P}_p(t)$. This basis is called the* nodal basis *of t.*

Chứng minh. We first verify that ϕ_l are well defined by showing their existence and uniqueness. Assume $(\hat{i}/p, \hat{j}/p, \hat{k}/p)$ is the barycentric coordinates of $n_{\hat{l}}$. Let P be the polynomial of degree p defined as follows

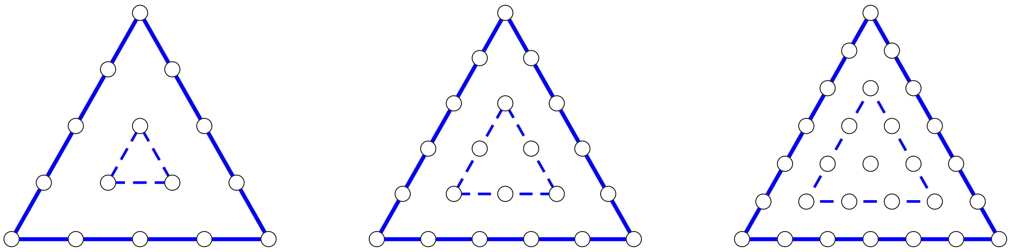
$$P = \prod_{i=0}^{\hat{i}-1}\left(c_1 - \frac{i}{p}\right)\prod_{j=0}^{\hat{j}-1}\left(c_2 - \frac{j}{p}\right)\prod_{k=0}^{\hat{k}-1}\left(c_3 - \frac{k}{p}\right).$$

Clearly, P is of degree p and is nonzero at $n_{\hat{l}}$. Now we consider a different nodal point n_l which is also of degree p and has barycentric coordinates $(i/p, j/p, k/p)$. Since $i + j + k = p = \hat{i} + \hat{j} + \hat{k}$, either $i < \hat{i}$ or $j < \hat{j}$ or $k < \hat{k}$. Without loss of generality, we can assume that $i < \hat{i}$. Then the formula of P contains the factor $c_1 - i/p$. This implies that P equals zero at n_l. Therefore, P is of degree p and vanishes at all of the nodal points of degree p except for $n_{\hat{l}}$. Consequently, $\phi_{\hat{l}}$ exists and can be written as $k_{\hat{l}}P$, where $k_{\hat{l}}$ is chosen so that $\phi_{\hat{l}}$ equals 1 at $n_{\hat{l}}$.

The uniqueness of ϕ_l comes from Lemma 3.2. Assume that ϕ'_l is another polynomial of degree p that equals 1 at n_l and zero at all other nodal points of degree p. Then $P = \phi_l - \phi'_l$ is a polynomial of degree p (or less) and P vanishes at all of the nodal points of degree p of t. By Lemma 3.2, $P \equiv 0$. Hence, $\phi_l \equiv \phi'_l$.

It remains to show that $\{\phi_l\}_{l=1}^{N_p}$ is actually a basis of $\mathscr{P}_p(t)$. Assume that the zero polynomial can be written as a linear combination of ϕ_l, i.e. $\sum_{l=1}^{N_p} \alpha_l \phi_l \equiv 0$. Evaluating both sides of this identity at nodal points of t, we have $\alpha_l = 0$ for all l. This implies that $\{\phi_l\}_{l=1}^{N_p}$ is a linearly independent set. On the other hand, the dimension of $\mathscr{P}_p(t)$ is N_p. Therefore, $\{\phi_l\}_{l=1}^{N_p}$ is a basis of $\mathscr{P}_p(t)$.

A nodal basis function can be referred to as



Hình 3. The element t' formed by interior nodal points of elements of degree $p = 4,5,6$.

a vertex, edge, or interior nodal basis function depending on the nodal point associated with it. However, in practice, they are usually called hat functions, bump functions and bubble functions respectively due to their shapes.

Corollary 3.4. *The following statements hold*

(i) A vertex basis function equals zero on the opposite edge.

(ii) An edge basis function equals zero on the other two edges.

(iii) An interior basis function equals zero on all edges.

Chứng minh. The proof of this corollary follows from the fact (shown in the proof of Theorem 3.3) that the basis functions associated with nodal points $(\hat{i}/p, \hat{j}/p, \hat{k}/p)$ is uniquely determined by

$$\phi = k\prod_{i=0}^{\hat{i}-1}\left(c_1 - \frac{i}{p}\right)\prod_{j=0}^{\hat{j}-1}\left(c_2 - \frac{j}{p}\right)\prod_{k=0}^{\hat{k}-1}\left(c_3 - \frac{k}{p}\right),$$

where k is a constant.

Proposition 3.5. *Let e be the shared edge of two elements t and t' in the triangulation $\mathscr{T}$. If $P \in \mathscr{P}_p(t)$ and $Q \in \mathscr{P}_p(t)$ agree at all of the nodal points on e (including the two vertices), then P and Q agree along the whole e.*

Chứng minh. The edge e can be parametrized using one parameter θ. Let $R = P - Q$. Then $R|_e$ is a polynomial of degree p, in variable θ. In addition, $R|_e$ vanishes at $p+1$ different values of θ associated with $p+1$ nodal points on e. Hence, $R|_e \equiv 0$. In other words, P and Q agree along the whole edge e.

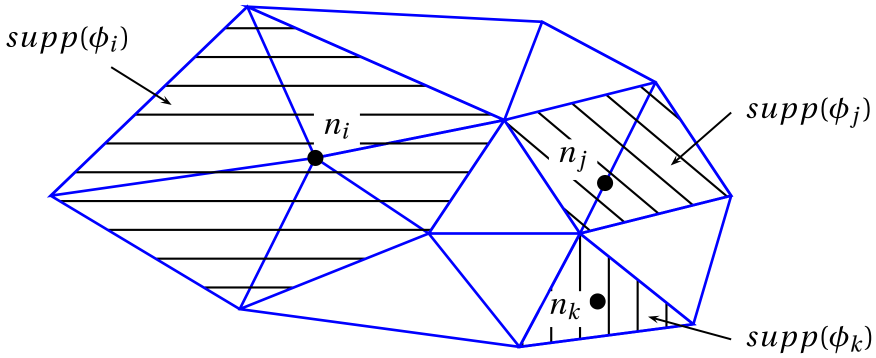
So far we have been focusing on basis functions defined on each element. Now we extend the definition to the whole triangulation.

Let $\mathscr{P}_p(\mathscr{T})$ be the space of C^0 (continuous) piecewise polynomials of degree p, namely, the space of continuous functions that are polynomials of degree p on each element of triangulation $\mathscr{T}$. Each element of $\mathscr{T}$ is equipped with a set of nodal points of degree p. Note that some of the vertex and edge nodal points are shared by more than one element. Similar to Theorem 3.3, we will define basis functions associated with these nodal points.

Theorem 3.6. *Consider a way of labeling the nodal points of the triangulation $\mathscr{T}$ from n_1 to n_N. Let ϕ_i be the C^0 piecewise polynomial of degree p defined on $\mathscr{T}$ that equals 1 at the nodal point n_i and equal 0 at all other nodal points of $\mathscr{T}$. Then $\{\phi_i\}_{i=1}^N$ is a basis of $\mathscr{P}_p(\mathscr{T})$. This basis is called the* nodal basis *of $\mathscr{T}$.*

Chứng minh. We first verify that ϕ_i are well defined by showing their existence and uniqueness. It is sufficient to show that such ϕ_i are uniquely defined on each element and smooth along shared edges of elements since they are C^0 piecewise polynomials.

Let t be an element in $\mathscr{T}$. If n_i does not belong to t, then by definition ϕ_i should be zero at all of the nodal point of degree p of t. By Lemma 3.2, $\phi_i|_t \equiv 0$. If n_i does belong to t, then ϕ_i equals 1 at n_i and equals zero at all other nodal points of degree p of t. By Theorem 3.3, ϕ_i is the basis function of $\mathscr{P}_p(t)$ associated with the nodal point n_i.



Hình 4. Supports of different kinds of basis functions.

The smoothness (continuity) of ϕ_i along the shared edges of elements is obtained by using Proposition 3.5 and noting that two neighboring elements of the same degree share the same set of nodal points along the common edge.

It remains to show that $\{\phi_i\}_{i=1}^N$ is actually a basis of $\mathscr{P}_p(\mathscr{T})$. First, an argument similar to the one used in the proof of Theorem 3.3 shows that $\{\phi_i\}_{i=1}^N$ are linearly independent. Now let P be an arbitrary function in $\mathscr{P}_p(\mathscr{T})$. Second, we will show that P can be written as a linear combination of $\{\phi_i\}_{i=1}^N$. Let $P' = \sum_{i=1}^N c_i\phi_i$, where c_i is the value of P at nodal point n_i. Because $\{\phi_i\}_{i=1}^N$ are C^0 piecewise polynomial of degree p, so is P'. Furthermore, from definition of P', $P-P'$ equals zero at all of the nodal points of $\mathscr{T}$. By Lemma 3.2, $P-P'$ is zero on each element of $\mathscr{T}$. Therefore, $P-P'$ is zero on the whole triangulation $\mathscr{T}$. In other words, $P=\sum_{i=1}^N c_i\phi_i$. This completes our proof.

In the proof of Theorem 3.6, we observe that $\phi_i|_t \equiv 0$ for almost all elements $t \in \mathscr{T}$, except the ones that touch the nodal point n_i. In other words, these basis functions have compact support. Figure 4 illustrates three different kinds of support associated with different types of basis functions.

In finite element method, solution is sought as a linear combination of basis functions of finite element space. If the space of piecewise polynomials of degree p, $\mathscr{P}_p(\mathscr{T})$, equipped with nodal basis functions defined in Theorem 3.6 is chosen to be the finite element space, then the coefficients c_i in the expression of the finite element solution $f_{f.e} = \sum_{i=1}^N c_i\phi_i$ is actually an approximation of the exact solution at the nodal point n_i. Because of this, c_i are called *degree of freedom* and the number of nodal points in $\mathscr{T}$ is called *number of degree of freedom*. Sometimes, the term "degree of freedom" is also used to refer to nodal points in a triangulation.

References

[1] I. Babuška and B. Q. Guo. The *h-p* version of the finite element method for problems with nonhomogeneous essential boundary condition. *Comput. Methods Appl. Mech. Engrg.*, 74(1):1–28, 1989.

[2] Mark Ainsworth and Bill Senior. An adaptive refinement strategy for *hp*-finite element computations. In *Proceedings of the International Centre for Mathematical Sciences Conference on Grid Adaptation in Computational PDEs: Theory and Applications (Edinburgh, 1996)*, volume 26, pages 165–178, 1998.

[3] W. Gui and I. Babuška. The *h, p* and *h-p* versions of the finite element method in 1 dimension. II. The error analysis of the *h*- and *h-p* versions. *Numer. Math.*, 49(6):613–657, 1986.

[4] Ivo Babuška and Theofanis Strouboulis. *The finite element method and its reliability*. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 2001.

[5] Randolph E. Bank and Hieu Nguyen. Domain decomposition and *hp*-adaptive finite elements. In *Domain Decomposition Methods in Science and Engineering XIX*, Lecture Notes in Computational Science and Engineering. Springer-Verlag, 2011.

[6] William F. Mitchell and Marjorie A. McClain. A survey of *hp*-adaptive strategies for elliptic partial differential equations. In Theodore E. Simos, editor, *Recent Advances in Computational and Applied Mathematics*, pages 227–258. Springer Netherlands, 2011.