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Strong two-scale convergence for a two-dimensional case

Hội tụ hai-kích thước mạnh cho một trường hợp hai chiều

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Abstract

In this paper, we present definitions and some properties of the classical strong two-scale convergence for component-wise vector or matrix functions in a two-dimensional case.

Keywords: two-scale homogenization; strong two-scale convergence; two-dimensional

Tóm tắt

Trong bài báo này, chúng tôi trình bày các định nghĩa và một số tính chất của hội tụ hai-kích thước mạnh cổ điển cho các hàm vectơ hoặc ma trận trong một trường hợp hai chiều.

Từ khóa: đồng nhất hóa hai-kích thước; hội tụ hai-kích thước mạnh; hai chiều

1. Introduction

We are given in dimension two, a bounded reference domain $\Omega = \Omega^1 \times \Omega^2 \in \mathbb{R} \times \mathbb{R}$ and a variable $\boldsymbol{x} = (x^1, x^2) \in \Omega$. In two-scale homogenization theory, strong two-scale convergence can be viewed as an intermediate property between the usual (one-scale) weak and strong convergence.

In light of this spirit, we first give a neccesary review of the usual weak convergence in the Hilbert space $\boldsymbol{L}^2(\Omega)$ then the definitions and properties of the classical strong two-scale convergence for component-wise vector or matrix functions [1, 2], in a two-dimensional case.

2. Preliminaries

Latin indices are in the set $\{1,2\}$. The space of functions, vector fields in $\mathbb{R}^2$, and 2×2 matrix fields, defined over Ω are represented respectively by italic capitals (e.g. $L^2(\Omega)$), boldface Roman capitals (e.g. $\boldsymbol{V}$), and special Roman capitals (e.g. $\mathbb{S}$).

In the rest of this paper, we use the following notations [1]:

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- $Y := [0,1]^2$ is the reference periodic cell.
- $\boldsymbol{C}_0(\Omega)$ is the space of functions that vanish at infinity.
- $\boldsymbol{C}^\infty_{\text{per}}(Y)$ denotes the Y-periodic $\boldsymbol{C}^\infty$ vector-valued functions in $\mathbb{R}^2$. Here, Y-periodic means 1-periodic in each variable $y^i, i = 1,2$.
- The notation $\boldsymbol{H}^1_{\text{per}}(Y)$, as the closure for the $\boldsymbol{H}^1$-norm of $\boldsymbol{C}^\infty_{\text{per}}(Y)$, is the space of vector-valued functions $\boldsymbol{v} \in \boldsymbol{L}^2(Y)$ such that $\boldsymbol{v}(y)$ is Y-periodic in $\mathbb{R}^2$.
- $$\langle \boldsymbol{v} \rangle_y = \frac{1}{|Y|} \int_Y \boldsymbol{v}(y)\,\mathrm{d}y.$$
- $$\boldsymbol{H}_{\text{per}}(Y) := \{\boldsymbol{v} \in \boldsymbol{H}^1_{\text{per}}(Y) \,|\, \langle \boldsymbol{v} \rangle_y = 0\}.$$
- We use $\cdot$ for the canonical inner products in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively.
- The notation $\lesssim$ stands for $\leq$ up to a multiplicative constant that only depends on Ω when applicable.

The Sobolev norm $\|\cdot\|_{\boldsymbol{W}^{1,2}_0(\Omega)}$ has the form

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}};$$

here, $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} := \||\boldsymbol{v}|\|_{L^2(\Omega)}$, where $|\boldsymbol{v}|$ represents the Euclidean norm of the 2-component vector-valued function $\boldsymbol{v}$, and $\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla \boldsymbol{v}|\|_{\mathbb{L}^2(\Omega)}$, where $|\nabla \boldsymbol{v}|$ denotes the Frobenius norm of the 2×2 matrix $\nabla \boldsymbol{v}$. Recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ is specified by $|\boldsymbol{X}|^2 := \boldsymbol{X} \cdot \boldsymbol{X} = \text{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$.

Let ϵ be some natural small scale. For potential applications in homogenization, based on [3, 4, 5, 6], we consider $\boldsymbol{u}_\epsilon(\boldsymbol{x}) \in \boldsymbol{W}^{1,2}_0(\Omega)$ depending on x^1 only, that is, $\boldsymbol{u}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}_\epsilon(x^1)$, with boundary conditions of Neumann type. As remarked in [7], we do not distinguish between a function on $\mathbb{R}$ and its extension to $\mathbb{R}^2$ as a function of the first variable. It is assumed that $\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}\left(\frac{x^1}{\epsilon}\right)$ is a periodic function in x^1 with period ϵ, equivalently, $\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}(y^1)$ is a periodic function in y^1 with period 1. That is, for any integer k,

$$\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}_\epsilon(x^1 + \epsilon) = \boldsymbol{u}_\epsilon(x^1 + k\epsilon),$$

equivalently,

$$\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon} + 1\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon} + k1\right) = \boldsymbol{u}(y^1 + k).$$

3. Weak convergence

In the Hilbert space $\boldsymbol{L}^2(\Omega)$, we describe the basic notions of the usual weak convergence, which is defined below [8].

Consider a sequence of functions $\boldsymbol{u}_\epsilon \in \boldsymbol{L}^2(\Omega)$. Then, $(\boldsymbol{u}_\epsilon)$ is said to be bounded in $\boldsymbol{L}^2(\Omega)$ if

$$\limsup_{\epsilon\to 0} \int_\Omega |\boldsymbol{u}_\epsilon|^2\,\mathrm{d}x \leq c < \infty,$$

for some positive constant c.

By definition, a sequence $(\boldsymbol{u}_\epsilon(\boldsymbol{x})) \in \boldsymbol{L}^2(\Omega)$ is weakly convergent to $\boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{L}^2(\Omega)$ as $\epsilon \to 0$, denoted by $\boldsymbol{u}_\epsilon \rightharpoonup \boldsymbol{u}$, if

$$\lim_{\epsilon\to 0} \int_\Omega \boldsymbol{u}_\epsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}\,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\phi}\,\mathrm{d}x, \qquad (1)$$

for any test function $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$.

Furthermore, a sequence $(\boldsymbol{u}_\epsilon)$ in $\boldsymbol{L}^2(\Omega)$ is called strongly convergent to $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ as $\epsilon \to 0$, denoted by $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$, if

$$\lim_{\epsilon\to 0} \int_\Omega \boldsymbol{u}_\epsilon \cdot \boldsymbol{v}_\epsilon\,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x, \qquad (2)$$

for every sequence $(\boldsymbol{v}_\epsilon) \in \boldsymbol{L}^2(\Omega)$ which is weakly convergent to $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

The following are well-known weak convergence properties in $\boldsymbol{L}^2(\Omega)$.

(a) Any weakly convergent sequence is bounded in $\boldsymbol{L}^2(\Omega)$.

(b) Compactness principle: any bounded sequence in $\boldsymbol{L}^2(\Omega)$ has a weakly convergent subsequence.

(c) If a sequence $(\boldsymbol{u}_\epsilon)$ is bounded in $\boldsymbol{L}^2(\Omega)$ and (1) is satisfied for all $\boldsymbol{\phi} \in \boldsymbol{C}^\infty_0(\Omega)$, then $\boldsymbol{u}_\epsilon \rightharpoonup \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$.

(d) If $\boldsymbol{u}_\epsilon \to \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{v}_\epsilon \rightharpoonup \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, then

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon \cdot \boldsymbol{v}_\epsilon \,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x.$$

(e) Weak convergence of $(\boldsymbol{u}_\epsilon)$ to $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$ together with

$$\lim_{\epsilon\to 0}\int_\Omega |\boldsymbol{u}_\epsilon|^2\,\mathrm{d}x = \int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x$$

is equivalent to strong convergence of $(\boldsymbol{u}_\epsilon)$ to $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$.

Hereafter, we denote by $Y = [0,1]^2$ the cell of periodicity. (In our paper, a periodic cell has the form $Y = [0,1] \times [0,1]$.) The mean value of a 1-periodic function $\boldsymbol{\psi}(y^1)$ is denoted by $\langle\boldsymbol{\psi}\rangle$, that is,

$$\langle\boldsymbol{\psi}\rangle \equiv \int_{Y^1} \boldsymbol{\psi}(y^1)\,\mathrm{d}y^1.$$

Recall that $y^1 = \epsilon^{-1}x^1$, and we do not distinguish between a function on Y^1 and its extension to Y as a function of the first variable only.

Also, in our paper, the symbol $\boldsymbol{L}^2(Y)$ is used not only for functions defined on Y but also for the space of functions in $\boldsymbol{L}^2(Y)$ extended by 1-periodicity to all $\mathbb{R}^2$. Similarly, $\boldsymbol{C}^\infty_{\text{per}}(Y)$ represents the space of infinitely differentiable 1-periodic functions on the entire $\mathbb{R}^2$.

For later use, we need the following classical result.

Lemma 3.1 (**The mean value property**). *Let $\boldsymbol{h}(y^1)$ be a 1-periodic function on $\mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{L}^2(Y^1)$. Then, for any bounded domain Ω, there holds the weak convergence*

$$\boldsymbol{h}\left(\frac{x^1}{\epsilon}\right) \rightharpoonup \langle\boldsymbol{h}\rangle \;in\; \boldsymbol{L}^2(\Omega)\;\; as\;\; \epsilon\to 0. \tag{3}$$

Proof. The proof is based on property (c) and can be found in [8].

4. Weak two-scale convergence

We have the following definition of weak two-scale convergence in $L^2(\Omega)$ (introduced by in 1989 by Nguetseng) [1, 2].

Definition 4.1. *Let (u_ϵ) be a bounded sequence in $L^2(\Omega)$. If there exist a subsequence, still denoted by u_ϵ, and a function $u(\boldsymbol{x}, y^1) \in L^2(\Omega\times Y^1)$, where $Y^1 = [0,1]$ such that*

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega & u_\epsilon(\boldsymbol{x})\left(\phi(\boldsymbol{x})h\left(\frac{x^1}{\epsilon}\right)\right)\mathrm{d}x \\ &= \int_{\Omega\times Y^1} u(\boldsymbol{x},y^1)(\phi(\boldsymbol{x})h(y^1))\,\mathrm{d}x\,\mathrm{d}y^1\end{aligned} \tag{4}$$

for any $\phi \in C_0^\infty(\Omega)$ and any $h \in C^\infty_{per}(Y^1)$, then such a sequence u_ϵ is said to weakly two-scale converge to $u(\boldsymbol{x}, y^1)$. This convergence is denoted by $u_\epsilon(\boldsymbol{x}) \rightharpoonup u(\boldsymbol{x}, y^1)$.

For vectors $\boldsymbol{u}_\epsilon$, equation (4) implies

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega & \boldsymbol{u}_\epsilon(\boldsymbol{x})\cdot\boldsymbol{\Phi}\left(\boldsymbol{x},\frac{x^1}{\epsilon}\right)\mathrm{d}x \\ &= \int_{\Omega\times Y^1} \boldsymbol{u}(\boldsymbol{x},y^1)\cdot\boldsymbol{\Phi}(\boldsymbol{x},y^1)\,\mathrm{d}x\,\mathrm{d}y^1,\end{aligned} \tag{5}$$

for every $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega;\boldsymbol{C}_{\text{per}}(Y^1))$, whose choice is explained in [9] (p. 8).

5. Strong two-scale convergence

The further extension of the class of test functions in Definition 4.1 leads to the basis of the following notion of the classical strong two-scale convergence [8, 10].

Definition 5.1. *A bounded sequence $u_\epsilon \in L^2(\Omega)$ is called strongly two-scale convergent if there exists $u = u(\boldsymbol{x}, y^1) \in L^2(\Omega\times Y^1)$ such that*

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega & u_\epsilon(\boldsymbol{x})v_\epsilon(\boldsymbol{x})\,dx \\ &= \int_{\Omega\times Y^1} u(\boldsymbol{x},y^1)v(\boldsymbol{x},y^1)\,dx\,dy^1\end{aligned} \tag{6}$$

for any bounded sequence $v_\epsilon(\boldsymbol{x}) \in L^2(\Omega)$ such that $v_\epsilon(\boldsymbol{x}) \rightharpoonup v(\boldsymbol{x}, y^1) \in L^2(\Omega)$. This convergence is denoted by $u_\epsilon(\boldsymbol{x}) \twoheadrightarrow u(\boldsymbol{x}, y^1)$.

For vector (or matrix) $\boldsymbol{u}_\epsilon$, equation (6) implies

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega & \boldsymbol{u}_\epsilon(\boldsymbol{x})\cdot\boldsymbol{v}_\epsilon(\boldsymbol{x})\,dx \\ &= \int_{\Omega\times Y^1} \boldsymbol{u}(\boldsymbol{x},y^1)\cdot\boldsymbol{v}(\boldsymbol{x},y^1)\,dx\,dy^1.\end{aligned} \tag{7}$$

In the next well-known results, weak and strong two-scale convergence can be viewed as intermediate properties between the usual (one-scale) weak and strong convergence.

Proposition 5.2. *Let (u_ϵ) be a sequence in $L^2(\Omega)$ and $u \in L^2(\Omega \times Y^1)$. Then,*

(i) $u_\epsilon \to u$ *in* $L^2(\Omega) \implies u_\epsilon \twoheadrightarrow u$ *in* $L^2(\Omega \times Y^1)$,

whenever u is independent of y^1, the converse also holds,

(ii) $u_\epsilon \twoheadrightarrow$ *in* $L^2(\Omega \times Y^1) \implies u_\epsilon \rightharpoonup u$ *in* $L^2(\Omega \times Y^1)$,

(iii) $u_\epsilon \rightharpoonup u$ *in* $L^2(\Omega \times Y^1) \implies u_\epsilon \rightharpoonup \int_{Y^1} u(\cdot, y^1)\, dy^1$ *in* $L^2(\Omega)$.

Proof. For (i), the proof is readily followed from Definition 4.1, the mean value property (3), and the property (d) of convergence in L^2.

For (ii), it is obvious. Indeed, it suffices to take, in Definition 5.1,

$$v_\epsilon(\boldsymbol{x}) = \phi(\boldsymbol{x}) h(\epsilon^{-1} x^1),$$

$\phi \in C_0^\infty(\Omega), h \in L^2(Y^1)$, and recall (3), to derive (4) as desired. Moreover, from $u_\epsilon \twoheadrightarrow u$ in $L^2(\Omega \times Y^1)$ (6), taking $v_\epsilon = u_\epsilon$, one obtains the relation

$$\lim_{\epsilon\to 0} \int_\Omega |u_\epsilon(\boldsymbol{x})|^2\, dx = \int_{\Omega\times Y^1} |u(\boldsymbol{x}, y^1)|^2\, dx\, dy^1. \tag{8}$$

For (iii), by the definition of weak two-scale convergence 4.1, it follows that

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(\boldsymbol{x}) \Phi\left(\boldsymbol{x}, \frac{x^1}{\epsilon}\right) dx = \int_{\Omega\times Y^1} u(\boldsymbol{x}, y^1) \Phi(\boldsymbol{x}, y^1)\, dx\, dy^1, \tag{9}$$

for every $\Phi \in L^2(\Omega; C_{\text{per}}(Y^1))$. Choosing $\Phi = 1$ in (9) and applying the property (c) of weak convergence, one obtains

$$u_\epsilon(\boldsymbol{x}) \rightharpoonup \int_{Y^1} u(\boldsymbol{x}, y^1)\, dy^1 = \langle u(\boldsymbol{x}, \cdot)\rangle, \tag{10}$$

which implies that one can reach the usual weak limit from the two-scale limit by taking the average over the cell of periodicity.

The converse of (ii) is also true as follows [8].

Lemma 5.3. *Weak two-scale convergence $u_\epsilon(\boldsymbol{x}) \rightharpoonup u(\boldsymbol{x}, y^1)$ together with the relation (8) implies strong two-scale convergence $u_\epsilon(\boldsymbol{x}) \twoheadrightarrow u(\boldsymbol{x}, y^1)$.*

Proof. The proof is based on [8]. Consider an arbitrary subsequence (still denoted by ϵ) $\epsilon \to 0$ such that there exist limits

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(\boldsymbol{x}) v_\epsilon(\boldsymbol{x})\, dx = \alpha, \ \lim_{\epsilon\to 0} \int_\Omega |v_\epsilon(\boldsymbol{x})|^2\, dx = \beta,$$

where $v_\epsilon(\boldsymbol{x}) \rightharpoonup v(\boldsymbol{x}, y^1)$. Then, using the lower semicontinuity property [8] for $t v_\epsilon + u_\epsilon$, we obtain

$$\lim_{\epsilon\to 0} \int_\Omega |t v_\epsilon(\boldsymbol{x}) + u_\epsilon(\boldsymbol{x})|^2\, dx \geq \int_{\Omega\times Y^1} |t v(\boldsymbol{x}, y^1) + u(\boldsymbol{x}, y^1)|^2\, dx dy^1.$$

Applying (8), we get

$$t^2\beta + 2t\alpha \geq t^2 \int_{\Omega\times Y^1} |v|^2 dx\, dy^1 + 2t \int_{\Omega\times Y^1} uv\, dx\, dy^1.$$

Hence,

$$2t\left(\alpha - \int_{\Omega\times Y^1} uv\, dx\, dy^1\right) \geq t^2\left(-\beta + \int_{\Omega\times Y^1} |v|^2 dx\, dy^1\right).$$

On the right hand side of this inequality, we apply the lower semicontinuity property [8] again for $\boldsymbol{v}_\epsilon$. Then, with the arbitrariness of t, we must have

$$\alpha = \int_{\Omega\times Y^1} uv\, dx\, dy^1,$$

which is our desired result.

The following theorem is stated and proved in [8].

Theorem 5.4. *Let $u_\epsilon(\boldsymbol{x}) \in L^2(\Omega)$, $u_\epsilon(\boldsymbol{x}) \twoheadrightarrow u(\boldsymbol{x}, y^1)$. Suppose in addition that $u(\boldsymbol{x}, y^1)$ is a Carathéodory function, $u(\boldsymbol{x}, y^1) \leq \Phi_0(y^1)$, $\Phi_0 \in L^2(Y^1)$. Then,*

$$\lim_{\epsilon\to 0} \int_\Omega |u_\epsilon(\boldsymbol{x}) - u(\boldsymbol{x}, \epsilon^{-1} x^1)|^2 dx = 0. \tag{11}$$

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